Extensions of GR using Projective-Invariance

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Abstract. We show that the unification of electromagnetism and gravity into a single geometrical entity can be beautifully accomplished in a theory with non-symmetric affine connection ($\Gamma^{\lambda}_{\mu\nu} \neq \Gamma^{\lambda}_{\nu\mu}$), and the unifying symmetry being projective symmetry. In addition, we show that in a purely-affine theory where there are no constrains on the symmetry of $\Gamma^{\lambda}_{\mu\nu}$, the electromagnetic field can be interpreted as the field that preserves projective-invariance. The matter Lagrangian breaks the projective-invariance, generating classical relativistic gravity and quantum electromagnetism. We notice that, if we associate the electromagnetic field tensor with the second Ricci tensor and $\Gamma^{\nu}_{[\mu\nu]}$ with the vector potential, then the classical Einstein-Maxwell equation can be obtained. In addition, we explain the geometrical interpretation of projective transformations. Finally, we discuss the importance of the role of projective-invariance in f(R) gravity theories.

Keywords: Extensions of General Relativity, Projective-Invariance, Second Ricci tensor

1. Introduction

Classical unified theories are considered well-meaning topics to explore. Historically, they gave physicists a clue for finding a unified field theory since a classical unified theory can be viewed as the classical limit of a quantum unified theory e.g. Maxwell unified theory of electricity and magnetism is the classical limit of Quantum Electrodynamics, therefore knowing how to combine gravity and electromagnetism could give us some insights when quantizing the gravitational field.

In general relativity, the electromagnetic fields and matter fields are considered to be on the side of the matter tensor in the field equations, i.e. they act as sources of the gravitational field. In unified theory, the electromagnetic field must obtains the same geometric status as the gravitational field. A general affine connection not restricted to be symmetric has enough degrees of freedom to make it possible to describe the classical gravitational and electromagnetic fields. The general theory of relativity relates gravitational effects to the curvature of space. The electromagnetic tensor has classically been introduced separately from geometry, the electromagnetic stress energy tensor acting as a source of the gravitational field, thus while gravitation has been expressed as purely geometrical theory, electromagnetism has been coupled to geometry but with the presence of an additional non-geometrical element, the electromagnetic tensor, for its description.

Theories based on projective-invariance work by formulating a Lagrangian that is projectively invariant. However, this Lagrangian doesn't determine the connection completely because this Lagrangian is invariant under these projective transformations, so we have add to a term that breaks this invariance, such as the second Ricci tensor:

$$Q_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\rho\nu,\mu} - \Gamma^{\rho}_{\rho\mu,\nu}. \tag{1}$$

Under the projective transformation $\Gamma^{\rho}_{\mu\nu} \mapsto \Gamma^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\Lambda_{\nu}$, the tensor $Q_{\mu\nu}$ changes to $Q_{\mu\nu} \mapsto Q_{\mu\nu} + 4(\Lambda_{\nu,\mu} - \Lambda_{\mu,\nu})$, which breaks the projective-invariance. Another way to break the projective-invariance is to constrain the way the connection enters our lagrangian using a Lagrange multiplier. Imposing constrains on our lagrangian is a more natural way of deriving the field equations and determining the connection completely than adding extra terms to the lagrangian that have no well-meaning physical interpretation.

In the following sections we discuss the different types of projective transformations and show how these transformations can be used to

formulate a purely-affine theory that can incorporate both the gravitational and electromagnetic field in one set of equations. We also discuss the role of projective-invariance in deriving field equations for a lagrangian that is linear in the Ricci scalar and replacing the unphysical constrains on the forms of matter that can enter our lagrangian.

2. Two types of projective transformations

Here we will discuss two types of projective symmetries:

TYPE: I

$$\Gamma^{\rho}_{\mu\nu} \mapsto \hat{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\lambda_{,\nu}, \tag{2}$$

where λ is some undetermined function.

Under this type of transformation the curvature tensor $R^{\rho}_{\mu\sigma\nu} = \Gamma^{\rho}_{\mu\nu,\sigma} + \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\kappa}_{\mu\nu}\Gamma^{\rho}_{\kappa\sigma} - \Gamma^{\kappa}_{\mu\sigma}\Gamma^{\rho}_{\kappa\nu}$ is invariant.

Consider the invariant Einstein-Hilbert (Feynman, 2003) Lagrangian density:

$$\mathcal{L}_g = -\frac{1}{2\kappa}R_{\mu\nu}\mathrm{g}^{\mu\nu}, \tag{3}$$

where $\kappa = 8\pi G(c=1)$, $\mathrm{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$(the fundamental tensor density) and $g = det(g_{\mu\nu})$. The total Lagrangian density for gravitational and matter fields is given by $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$. Consequently, a theory characterized by $R_{\mu\nu}$ cannot determine the Γ-field but only up to an arbitrary function λ hence in this theory $\Gamma^{\rho}_{\mu\nu}$ and $\hat{\Gamma}^{\rho}_{\mu\nu}$ represent the same field, but this λ-transformation produces a non-symmetric $\hat{\Gamma}$-field form a symmetric Γ-field, hence the symmetry condition for the Γ-field loses objective significance. To make the calculation easier, we replace $R_{\mu\nu}$ by the transposition invariant tensor $E_{\mu\nu}$:

$$E_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} + \frac{\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\nu\sigma,\mu}}{2} + \Gamma^{\kappa}_{\mu\nu}\Gamma^{\sigma}_{\kappa\sigma} - \Gamma^{\kappa}_{\mu\sigma}\Gamma^{\sigma}_{\kappa\nu}. \tag{4}$$

Similar to $R_{\mu\nu}$, $E_{\mu\nu}$ is also invariant under this type of transformation. Separating the symmetric and antisymmetric parts of $E_{\mu\nu}$ and varying the lagrangian with respect to the symmetric and antisymmetric parts of the metric. We get the field equations

$$R_{\mu\nu} - G^{\lambda}_{\mu\xi}G^{\xi}_{\lambda\nu} + \frac{1}{3}W_{\mu}W_{\nu} + \Gamma^{\lambda}_{(\mu\nu)}W_{\lambda} = 0 \tag{5}$$

$$G^{\lambda}_{\mu\nu;\lambda} - \frac{1}{3}(W_{\mu,\nu} - W_{\nu,\mu}) = 0, \tag{6}$$

where

$$W_\mu = \frac{1}{2}(\Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\nu\mu}) \tag{7}$$

$$G^\lambda_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} + \frac{1}{3}\delta^\lambda_\mu W_\nu - \frac{1}{3}\delta^\lambda_\nu \Gamma_\mu, \tag{8}$$

and $G^\lambda_{\mu\nu;\lambda}$ is the covariant derivative with respect to the symmetric $\Gamma^\lambda_{(\mu\nu)}$. Notice that W_μ is the common source for (5) and (6), however in a theory where W_μ vanishes like Einstein's theory of gravitation, the two equations have no common source, therefore it would be impossible to determine whether these two equations came form the variation of the same lagrangian. It seems natural to think that this symmetry if exists may of broken in the early universe when matter emerged from the radiation universe. As a result, now we have two separate fields of gravitation and electromagnetism. The symmetry of the connection is a characteristic of Riemann geometry and Einstein's theory of gravitation. However, for a theory that consists of the electromagnetic field too, the condition of symmetry can be relaxed. We can identify W_μ with the electromagnetic vector potential A_μ.

We can determine the equations of the connection with a Lagrange multiplier B^μ multiplied by $G^\lambda_{\mu\lambda}$, such that (S.N. Bose, 1953)

$$\delta \int d^4x(\kappa\mathcal{L} - 2B^\mu G^\lambda_{\mu\nu}) = 0. \tag{9}$$

We obtain the following equation for the connection

$$g^{[\mu\nu]}{}_{,\nu} = \sqrt{|g|}(g^{(\mu\rho)}\Gamma_\rho + \frac{3}{2}g^{(\sigma\lambda)}\Gamma^\mu_{(\sigma\lambda)}) \tag{10}$$

and

$$B^\mu_{,\mu} = 0. \tag{11}$$

However, if we use a different Lagrange multiplier with constrain $S_\mu = 0$, where S_μ is the trace of the torsion tensor, such that

$$\delta S = \int d^4x(\mathcal{L} - g^{[\mu\nu]}B^\rho_{\mu\nu}S_\rho) = 0, \tag{12}$$

then, its easy to show that

$$\lambda_{,\mu} = \frac{1}{3}(\Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\nu\mu}). \tag{13}$$

Type: ii

$$\Gamma^\rho_{\mu\nu} \mapsto \hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu \Lambda_\nu \tag{14}$$

Under this transformation the Ricci scalar remains invariant (i.e the gravitation action $\frac{1}{2\kappa}\int d^4x\sqrt{-g}R$ is projectively invariant). The simplest lagrangian density to adopt in a affine field theory is the square root of the ricci tensor. The condition for a lagrangian density to be covariant is that it must be a product of a scalar and the square root of the determinant of a covariant tensor. It is easy to show that this leads to reasonable result. Consider the lagrangian density

$$\mathcal{L} = -\frac{2}{\Lambda}\sqrt{-R_{\mu\nu}}, \tag{15}$$

By varying the Ricci tensor using the Platini formula (Schrödinger, 1945; Schrödinger, 1948) and using the definition of the fundamental tensor $\mathrm{g}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial R_{\mu\nu}}$, the variation of (15) yields (Schrödinger, 1945)

$$\begin{aligned}
\delta S &= \int d^4x(-\mathrm{g}^{\mu\nu}_{;\rho}\delta\Gamma^{\rho}_{\mu\nu} + \mathrm{g}^{\mu\nu}_{;\nu}\delta\Gamma^{\rho}_{\mu\rho} + 2\mathrm{g}^{\mu\nu}S_\rho\delta\Gamma^{\rho}_{\mu\nu} \\
&-2\mathrm{g}^{\mu\nu}S_\nu\delta\Gamma^{\rho}_{\mu\rho} - 2\mathrm{g}^{\mu\nu}S^{\sigma}_{\rho\nu}\delta\Gamma^{\rho}_{\mu\sigma}) \\
&= \int d^4x\delta\Gamma^{\rho}_{\mu\nu}(-\mathrm{g}^{\mu\nu}_{;\rho} + \mathrm{g}^{\mu\sigma}_{;\sigma}\delta^{\rho}_{\nu} + 2\mathrm{g}^{\mu\nu}S_\rho \\
&-2\mathrm{g}^{\mu\sigma}S_\sigma\delta^{\rho}_{\nu} - 2\mathrm{g}^{\mu\sigma}S^{\nu}_{\rho\sigma}) \\
&= 0,
\end{aligned} \tag{16}$$

Here, we have used the identity (Schrödinger, 1950)

$$\int d^4x(\sqrt{-g}V^\mu)_{;\mu} = 2\int d^4x(\sqrt{-g}S_\mu V^\mu) \tag{17}$$

For an arbitrary variation $\delta\Gamma^{\rho}_{\mu\nu}$ this gives:

$$\mathrm{g}^{\mu\nu}_{;\rho} - \mathrm{g}^{\mu\sigma}_{;\sigma}\delta^{\rho}_{\nu} - 2\mathrm{g}^{\mu\nu}S_\rho + 2\mathrm{g}^{\mu\sigma}S_\sigma\delta^{\nu}_{\rho} + 2\mathrm{g}^{\mu\sigma}S^{\nu}_{\rho\sigma} = 0. \tag{18}$$

Under a transformation of Type II where Λ_ν is replaced by $\frac{2}{3}W_\nu$, this becomes

$$\mathrm{g}^{\mu\nu}_{,\rho} + \mathrm{g}^{\sigma\nu}\hat{\Gamma}^{\mu}_{\sigma\rho} + \mathrm{g}^{\mu\sigma}\hat{\Gamma}^{\nu}_{\rho\sigma} - \frac{1}{2}(\hat{\Gamma}^{\sigma}_{\rho\sigma} + \hat{\Gamma}^{\sigma}_{\sigma\rho})\mathrm{g}^{\mu\nu} = 0 \tag{19}$$

$$\mathrm{g}^{[\mu\nu]}{}_{,\nu} - \frac{1}{2}(\hat{\Gamma}^{\rho}_{\rho\nu} - \hat{\Gamma}^{\rho}_{\nu\rho})\mathrm{g}^{(\mu\nu)} = 0 \tag{20}$$

By using the definition of $\hat{\Gamma}^{\rho}_{\nu\mu}$ (i.e. $\hat{\Gamma}^{\rho}_{\nu\rho} = \hat{\Gamma}^{\rho}_{\rho\nu}$) and contracting (19) with respect to (μ,ρ) then with respect to (ν,ρ) and subtracting the two resulting equations, we get our first field equation

$$\mathrm{g}^{[\mu\nu]}{}_{,\nu} = 0. \tag{21}$$

Although equation (19) is not the covariant derivative of $\mathrm{g}^{\mu\nu}$ with respect to $\hat{\Gamma}$, the reversal of the order of indices in the third term allows us to determine $\hat{\Gamma}$ uniquely. In addition, with the condition $(\hat{\Gamma}^{\rho}_{\nu\rho} = \hat{\Gamma}^{\rho}_{\rho\nu})$, $\hat{\Gamma}$ is reduced from 64 to 60 independent components. By using the identity (17) and $g_{\mu\nu}g^{\mu\nu} = 4$, one can show that

$$\hat{\Gamma}^{\sigma}_{\alpha\sigma} = \frac{\partial log(\sqrt{-\mathrm{g}})}{x_{\alpha}} \tag{22}$$

Under the transformation $\Gamma^{\rho}_{\mu\nu} \mapsto \Gamma^{\rho}_{\mu\nu} + \frac{2}{3}\delta^{\rho}_{\mu}W_{\nu}$ the Ricci tensor transforms like $R_{\mu\nu} \mapsto R_{\mu\nu} + \frac{2}{3}(W_{\mu,\nu} - W_{\nu,\mu})$. Again, if we identify W_{ν} with vector potential A_{ν} and $W_{\mu,\nu} - W_{\nu,\mu}$ with the electromagnetic field tensor $F_{\mu\nu}$, we can easily see how these two types of projective symmetries offer an attractive way of unifying the classical electromagnetic equations with general relativistic equations.

3. Unified theory from projective-invariance

It has been shown in (Ferraris et la., 1982) that the standard Einstein General Relativity is equivalent to a theory where the following two constrains are imposed: (i) The connection Γ is torsionless (i.e $\Gamma^{\alpha}_{\beta\mu} = \Gamma^{\alpha}_{\mu\beta}$) (ii) The lagrangian density depends only on the symmetric part of the Riemann tensor $R_{(\mu\nu)} = \frac{1}{2}(R^{\alpha}_{\mu\alpha\nu} + R^{\alpha}_{\nu\alpha\mu})$. A natural question arises concerning the unification of the different physical interactions of nature: Whether it is possible to describe the physical interaction of nature within the same framework?. It turns out that a unification of gravity with other physical interaction may be obtained in this context if and only if the constrains (i) and/or (ii) are relaxed. Here we will consider a lagrangian density based on a non-symmetric connection and the second Ricci tensor $Q_{\nu\mu} = R^{\alpha}_{\alpha\mu\nu}$ with electromagnetic tensor. If we consider the action $S = \int d^4xL_s(L, R, Q)$, the variation of this action gives:

$$\begin{aligned} \delta S &= \int d^4xL_s(L, R, Q) \\ &= \delta \int d^4x(\frac{\partial L_s}{\partial R_{\mu\nu}}\delta R_{\mu\nu} + \frac{\partial L_s}{\partial Q_{\mu\nu}}\delta Q_{\mu\nu} + \frac{\partial L_s}{\partial \Gamma^{\rho}_{\mu\nu}}\delta\Gamma^{\rho}_{\mu\nu}). \end{aligned} \tag{23}$$

Using the equation $\delta R_{\mu\nu} = \delta\Gamma^{\rho}_{\mu\nu;\rho} - \delta\Gamma^{\rho}_{\mu\rho;\nu} - 2S^{\sigma}_{\rho\nu}\delta\Gamma^{\rho}_{\mu\sigma}$, the identity (17) and the principle of least action, equation (18) changes to:

$$\begin{aligned} &\mathrm{g}^{\mu\nu}_{;\rho} - \mathrm{g}^{\mu\sigma}_{;\sigma}\delta^{\rho}_{\nu} - 2\mathrm{g}^{\mu\nu}S_{\rho} + 2\mathrm{g}^{\mu\sigma}S_{\sigma}\delta^{\nu}_{\rho} + 2\mathrm{g}^{\mu\sigma}S^{\nu}_{\rho\sigma} \\ &= 2M^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\rho} + N^{\mu\nu}_{\rho}, \end{aligned} \tag{24}$$

where

$$N^{\mu\nu}_{\rho} = \frac{\partial L_s}{\partial \Gamma^{\rho}_{\mu\nu}}, \tag{25}$$

$$M^{\mu\nu} = \frac{\partial L_s}{\partial Q_{\mu\nu}}. \tag{26}$$

This equation can be further simplified to

$$\mathrm{g}^{\mu\nu}_{,\rho} + \hat{\Gamma}^{\mu}_{\sigma\rho} g^{\sigma\nu} + \hat{\Gamma}^{\nu}_{\rho\sigma} g^{\mu\sigma} - \hat{\Gamma}^{\sigma}_{\sigma\rho} g^{\mu\nu} = N^{\mu\nu}_{\rho} - \frac{1}{3} N^{\mu\sigma}_{\sigma} \delta^{\mu}_{\nu} + 2M^{\nu\sigma}{}_{,\sigma} \delta^{\mu}_{\rho} - \frac{2}{3} M^{\mu\sigma}{}_{,\sigma} \delta^{\nu}_{\rho},$$

where $\hat{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\mu}_{\sigma\rho} + \frac{2}{3}\delta^{\rho}_{\mu} S_{\nu}$. Contracting the indices μ and ρ and assuming that the metric tensor is symmetric (i.e. $\frac{\partial L_s}{\partial R_{[\mu\nu]}} = 0$) we obtain

$$M^{\sigma\nu}{}_{,\sigma} = \frac{1}{8} N^{\sigma\nu}_{\sigma} \tag{27}$$

Although the symmetric Ricci tensor $R_{(\mu\nu)}$ and $\mathrm{g}^{\mu\nu}$ are invariant under a transformation of Type II, the second Ricci tensor changes according to $Q_{\mu\nu} \mapsto Q_{\mu\nu} + 4(\Lambda_{\nu,\mu} - \Lambda_{\mu,\nu})$. However, the total action $\int d^4x (8M^{\sigma\nu}{}_{,\sigma} + N^{\sigma\nu}_{\sigma}) \delta V_{\mu}$ is projectively invariant. If we use the antisymmetry of $M^{\sigma\nu}$ and associate $M^{\sigma\nu}_{,\sigma}$ with the electromagnetic vector density j^{ν} and notice the conservation of this vector density $j^{\nu}_{;\nu} = 0$, then we can interpret the electromagnetic field in this theory as the field that preserves the projective-invariance of this lagrangian.

4. Geometrical point of view

According to the theory of special relativity, light has a constant velocity of propagation. If a light ray travels from point (x_1, x_2, x_3, x_4) to $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$, then we can write the relation

$$dx_1^2 + dx_2^2 + dx_3^2 - c^2 dx_4^2 = 0 \tag{28}$$

or, more generally

$$\sum_{\mu\nu} \mathrm{g}_{\mu\nu} dx_{\mu} dx_{\nu} = 0 \tag{29}$$

where $\mathrm{g}_{\mu\nu}$ transforms in a definite way if a certain continuous coordinate tranformation is applied. Mathematically speaking, they are the components of a tensor with a property of symmetry (i.e. $\mathrm{g}_{\mu\nu} = \mathrm{g}_{\nu\mu}$). However, when considering extensions of gravity we must relax this

condition of symmetry, but then generally (28) doesn't hold. A more general option is to consider $\Gamma^{\rho}_{\mu\nu}$ to be a more fundamental object and consider a transformation that preserves the symmetry of $g_{\mu\nu}$ and breaks the symmetry of $\Gamma^{\rho}_{\mu\nu}$ with respect to permutations of lower indices.

The desire to eliminate the difference in the geometrical interpretations between the gravitational and electromagnetical fields has perhaps been one of the main motivation for looking for a generalization of General Relativity. Most of these attempts have involved a generalization of riemannien geometry.

General Relativity assumes the torsion-free condition. However, if we define a projective transformation as a transformation preformed on the torsion tensor (i.e $S^{\rho}_{\mu\nu} = 0 \neq \hat{S}^{\rho}_{\mu\nu}$), then we can interpret a projective transformation as the transformation that relaxes the condition of symmetry for the connection (Hehl, 1973).

If we consider (23) again and restrict the torsion tensor to be traceless ($S_{\mu} = 0$), then (27) become a stronger condition on how the lagrangian dependent on the condition. This condition enters the Lagrangian density as Lagrange multiplier term $-\frac{1}{2}D^{\mu}S_{\mu}$ where the Lagrange multiplier D^{μ} is a vector density. Consequently, equation (27) becomes ${M^{\sigma\nu}}_{,\sigma} = \frac{1}{8}N^{\sigma\nu}_{\sigma} - \frac{3}{16}D^{\nu}$. Setting $N^{\sigma\nu}_{\sigma} = \frac{3}{2}D^{\nu}$ yields the wave equation ${M^{\sigma\nu}}_{,\sigma} = 0$. However imposing this condition, j_{μ} need not be conserved(Hehl, 1985). Therefore relaxing the condition of symmetry and letting the action depend on $Q_{\mu\nu}$ gives a more suitable condition for unifying gravitation and electromagnetism than imposing $S_{\mu} = 0$.

5. $f(R)$ gravity and projective-invariance

f(R) gravity is a family of theories that try to modify or generalize Einstein's General theory of relativity, each defined with a different function of the Ricci scalar. The simplest of these theories is the Einstein-Hilbert action where the function is equal to the Ricci scalar. f(R) gravity can be used to produce a wide range of phenomena by adopting different functions (Sotiriou, 2006). Recently these theories have been used extensively to attack the problem of dark energy, since this phenomena is not predicted by General relativity. Historically, f(R) gravity was born to explain the simplicity of the gravitational action and whether its possible to modify this action to include enough information about the current structure of the universe.

The action

$$S = \int d^4x\sqrt{-g}f(R) \tag{30}$$

is projectively invariant under the projective transformation $\Gamma^{\rho}_{\mu\nu} \mapsto \Gamma^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\Lambda_{\nu}$ (i.e $\hat{R} = R$). Consequently, to be able to derive a consistent field equations we must find a way to break this invariance. One way to break this invariance is add terms to the action that are not projectively invariant, such as the homothetic curvature ($Q_{\mu\nu}$), another way is to constrain the connection to be symmetric (i.e. $\Gamma^{\rho}_{[\mu\nu]} = 0$).

Here we will consider the metric-affine formalism. Similar to the Platini formulism, metric-affine formalism considers the metric and the connection to be independent variables. However, the Platini formulism differs in the fact that it demands the matter action to be independent of the connection. The variation of the gravitational action gives

$$\begin{aligned} \delta S_g &= \frac{1}{2\kappa}\int d^4x\delta(\sqrt{-g}f(R)) \qquad (31) \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}(f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu})\delta g^{\mu\nu} + \\ &\frac{1}{2\kappa}\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}. \end{aligned}$$

If we allow the matter action to depend on the connection, then the variation of the mater action with respect to the metric and the connection gives

$$\delta S_m = \int d^4x(\frac{\partial S_m}{\partial g_{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial S_m}{\partial \Gamma^{\rho}_{\mu\nu}}\delta\Gamma^{\rho}_{\mu\nu}). \tag{32}$$

After taking the trace on μ and λ of equations (31) and (32), we get

$$-\frac{2\kappa}{\sqrt{-g}}\frac{\delta S_m}{\delta\Gamma^{\lambda}_{\mu\nu}} = 0. \tag{33}$$

This constrains how the connection enters the matter action. Consequently, the forms of matter that enter our matter action are restricted, which generally leads to inconsistences. Therefore, its clear that to get rid of these inconsistences, the form of (30) must be modified. One way to avoid these inconsistences is to add extra terms to (30). However, this procedure takes us away from our objective for explaining the simplicity of the gravitational action when considering f(R) gravity theories. It turns out that a more attractive way to overcome this

problem is to reconsider the property of projective-invariance of the gravitational action. Breaking the projective-invariance in this case allows to constrain the degrees of freedom and determine the way the connection enters out action. We can break the projective-invariance by adding a second term to the total action $\int d^4x Z^\mu S_\mu = 0$, where Z^μ is a Lagrange multiplier. After some tedious calculations we get $Z^\mu = -\frac{4}{3}\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta\Gamma^\lambda_{\lambda\mu}}$ and the resulting field equations are (Sotiriou, 2010)

$$S^\sigma_{\mu\sigma} = 0, \tag{34}$$

$$\begin{aligned}&\frac{1}{\sqrt{-g}}(\aleph^{\mu\nu}_{;\lambda} - \aleph^{\mu\sigma}_{;\sigma}\delta^\nu_\lambda) + 2\frac{\aleph^{\mu\sigma}}{\sqrt{-g}}S^\nu_{\sigma\lambda}\\ &= \kappa(\Delta^{\mu\nu}_\lambda - \frac{1}{3}(\Delta^{\sigma\nu}_\sigma\delta^\mu_\lambda - \Delta^{\sigma\mu}_\sigma\delta^\nu_\lambda)),\end{aligned} \tag{35}$$

where $\Delta^{\mu\nu}_\lambda = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta\Gamma^\lambda_{\mu\nu}}$ and $\aleph^{\mu\nu} = \sqrt{-g}f'(R)g^{\mu\nu}$.

Another way of breaking the projective-invariance is by imposing the condition $N_\nu = 0$, where $N_\nu = g^{\rho\sigma}g_{\rho\sigma;\nu}$. The resulting field equations are

$$N_\nu = 0, \tag{36}$$

$$\begin{aligned}&\frac{1}{\sqrt{-g}}(\aleph^{\mu\nu}_{;\lambda} - \aleph^{\mu\sigma}_{;\sigma}\delta^\nu_\lambda) + 2f'(R)g^{\mu\xi}(S^\sigma_{\lambda\sigma}\delta^\nu_\xi -\\ &S^\sigma_{\xi\sigma}\delta^\nu_\lambda + S^\nu_{\xi\lambda}) = \kappa(\Delta^{\mu\nu}_\lambda - \frac{1}{4}Z^\nu\delta^\mu_\lambda)).\end{aligned} \tag{37}$$

The trace of (37) gives $Z^\mu = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta\Gamma^\lambda_{\lambda\mu}}$.

Therefore, we have shown what Z^μ needs to be to solve these inconsistencies. It is clear that the procedure of projective-invariance presents an elegant way of solving these inconsistencies. However, as promising as this looks it lacks an element of generality. It easy to see that if we choose a different action (for example, a matter action linear in the connection), then this procedure is only valid if $f(R)$ is linear in R. However, if we are working with lagrangian that is linear in R, then the procedure of projective-invariance seems to be a natural way to use to derive the field equations.

6. Conclusions

In this paper we have shown that from the classical point of view of relativity field theory, a unification of the gravitational and electromagnetic fields can be beautifully achieved using a projective transformation. Moreover, in a theory where there are no constrains on the symmetry of the connection, if we associate W_μ with the electromagnetic vector potential A_μ, then with a projective transformation of Type II, we can achieve the Einstein-Maxwell equations. We also presented a geometrical point of view of the procedure of projective-invariance and demonstrated how imposing $S_\mu = 0$ prevents the conservation of j_μ. We also presented an interpretation of the role of the electromagnetic field in preserving projective transformations. Finally, we showed how the procedure of projective-invariance plays an important role in f(R) gravity theories, especially when deriving the field equations. The procedure of projective-invariance replaces the constrains on the forms of matter that enter our lagrangian with a more elegant looking field equations.

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